**Elliptic Filters**

The magnitude squared frequency response of the normalized low-pass elliptic filter of order $n$ is defined by

$$\left| H_n(j\omega) \right|^2 = \frac{1}{1 + \varepsilon^2 R_n^2(\omega)}$$

where $R_n(\omega)$ is a Chebyshev rational function of $\omega$ determined from the specified ripple characteristics.

![Graph of magnitude squared frequency response of elliptic LP filters of odd and even orders.](image)

**Figure 1:** Magnitude squared frequency response of elliptic LP filters of odd and even orders.

Unlike the Butterworth and Chebyshev filters, $\omega = 1$ is calculated using

$$\sqrt{\omega_1 \omega_2} = 1$$

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A parameter, \( \omega_r \), representing the sharpness of the transition region is defined as

\[
\omega_r = \frac{\omega_2}{\omega_1}
\]

Thus a large value of \( \omega_r \) indicates a large transition band, while a small value of \( \omega_r \) indicates a small transition band.

The general transfer function \( H_n(s) \), for the normalized low-pass elliptic filter is given from odd and even \( n \) by

\[
H_n(s) = \frac{H_0}{(s + s_0)} \prod_{i=1}^{n-1} \left( \frac{s^2 + A_{0i}}{s^2 + B_1_s + B_{0i}} \right), \text{ odd } n
\]

\[
H_n(s) = H_0 \prod_{i=1}^{n} \left( \frac{s^2 + A_{0i}}{s^2 + B_1_s + B_{0i}} \right), \text{ even } n
\]

To design a filter \( n,H_0,S_0,A_{0i},B_{1i},B_{0i} \) have to be determined from the design specifications

1. \( \varepsilon \)
2. \( A \)
3. \( \omega_r \)

or equivalently \( G_1,G_2 \) and \( \omega_r \), where

\[
G_1 = 20 \log \left[ \frac{1}{\sqrt{1 + \varepsilon^2}} \right] = 20 \log |H_n(j\omega_t)|
\]

\[
G_2 = 20 \log \left[ \frac{1}{A^2} \right] = 20 \log |H_n(j\omega_2)|
\]

By finding the \( G_1 \) and \( G_2 \) in this fashion, the \( \omega_r \) requirement will not be satisfied exactly; however, an \( \omega_r \) can be selected that exceeds the requirements.

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Determination of $n$ for Normalized Elliptic Filters

The design of a low-pass normalized elliptic filter to satisfy the specifications $G_1, G_2$ and $\omega_r$ using Table 3.6 is straightforward.

1. Find $G_1$ and $G_2$ in dB.
2. Find the $\omega_r$ portion of the table for which a value less than the determined $\omega_r$ is found.
3. Find the corresponding order $n$.
4. The values of $\omega_1$ and $\omega_2$ are obtained by

\[
\omega_1 = \frac{1}{\sqrt{\omega_r}} \quad \text{and} \quad \omega_2 = \sqrt{\omega_r}
\]

Thus it can be clearly verified that the requirements have been met.

The design of an un-normalized elliptic low-pass filter satisfying a $G_1$ dB ripple, cutoff at $\omega_1'$ and a $G_2$ dB gain at $\omega_2'$ can be obtained by LP -> LP transformation of a mutable normalized elliptic filter

\[
H(s) = H_{LP}(s) \bigg|_{s \rightarrow \frac{1}{\omega_0}}
\]

For the elliptic filter $\omega_0$ is selected to be the geometric mean of $\omega_1'$ and $\omega_2'$

\[
\omega_0 = \sqrt{\omega_1' \omega_2'}
\]

The corresponding LP requirements, $\omega_1, \omega_2$ and $\omega_r$ (transition) are obtained as follows

\[
\omega_1 = \frac{\omega_1'}{\omega_0}
\]
\[
\omega_2 = \frac{\omega_2'}{\omega_0}
\]
\[
\omega_r = \frac{\omega_2}{\omega_1} = \frac{\omega_2'}{\omega_1'}
\]

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Example #1

Find the transfer function for an elliptic low-pass filter with –2 dB cutoff value at 10,000 rad/s and a stop band attenuation of 40 dB for all \( \omega \) past 14,400 rad/s.

Solution

\[
\begin{align*}
\omega_1 &= 10,000 \text{ rad/s} \quad G_1 = -2 \text{ dB} \\
\omega_2 &= 14,400 \text{ rad/s} \quad G_2 = -40 \text{ dB}
\end{align*}
\]

Then

\[
\begin{align*}
\omega_0 &= \sqrt{\omega_2 \omega_1} = \sqrt{(1 \times 10^4)(1.44 \times 10^4)} = 12,000 \\
\omega_1 &= \frac{\omega_1}{\omega_0} = \frac{10,000}{12,000} = \frac{5}{6} \\
\omega_2 &= \frac{\omega_2}{\omega_0} = \frac{14,400}{12,000} = \frac{6}{5} \\
\omega_r &= \frac{\omega_2}{\omega_1} = \frac{\frac{6}{5}}{\frac{5}{6}} = 1.44
\end{align*}
\]

From the –2 dB and –40 dB part of the table it is seen that for \( n = 4 \) given \( \omega_r = 1.40542 \).

Thus

\[
H_{LP}(s) = \frac{0.01(s^2 + 7.25202)(s^2 + 1.57676)}{(s^2 + 0.467290s + 0.212344)(s^2 + 0.127954s + 0.677934)}
\]

The required LP is obtained by

\[
H(s) = H_{LP}(s) \bigg|_{s \to \frac{10,000}{12,000}}
\]

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Example #2

Find the transfer function \( H(s) \) for a normalized elliptic filter that will satisfy the following conditions:

\[
G_1 = -0.5\text{dB} \\
G_2 = -30\text{dB} \\
\omega_r = 1.21
\]

From table 3.6a with pass-band ripple of –0.5 dB and stop-band gain of –30 dB, the smallest value of \( n \) that satisfies the \( \omega_r \) is \( n = 5 \), giving \( \omega_r = 1.12912 \).

\[
H_5(s) = \frac{0.118807(s^2 + 2.14490)(s^2 + 1.18122)}{(s + 0.511701)(s^2 + 0.480774s + 0.648724)(s^2 + 0.088080s + 0.907216)}
\]

Where

\[
\omega_1 = \frac{1}{\sqrt{\omega_r}} = 0.941087 \\
\omega_2 = \sqrt{\omega_r} = \sqrt{1.12912} = 1.06260
\]